AD-781 800

OPTIMUM DETECTORS OF PSEUDONOISE WAVEFORMS

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Prepared for:

Naval Electronic Systems Command
Electronic Systems Director

19 June 1974

ESD-TR-74-256

AD781800

Technical Note 1974-37

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19 June 1974

Prepared for the Department of the Navy
under Electronic Systems Division Contract F19628-73-C-0002 by

Lincoln Laboratory
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
LEXINGTON, MASSACHUSETTS

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

AD-781 800

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER ESD-TR-74-225	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Optimum Detectors of Pseudonoise Waveforms		5. TYPE OF REPORT & PERIOD COVERED Technical Note
		6. PERFORMING ORG. REPORT NUMBER Technical Note 1974-37
7. AUTHOR(s) Bernstein, Steven L.		8. CONTRACT OR GRANT NUMBER(s) F19628-73-C-0002
9. PERFORMING ORGANIZATION NAME AND ADDRESS Lincoln Laboratory, M.I.T. P.O. Box 73 Lexington, MA 02173		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Project No. 1508A
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Electronic Systems Command Department of the Navy Washington, DC 20360		12. REPORT DATE 19 June 1974
		13. NUMBER OF PAGES 22
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Electronic Systems Division L.G. Hanscom Field Bedford, MA 01730		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

None

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

phase detectors Navy communications pseudonoise waveforms

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
LINCOLN LABORATORY

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S. L. BERNSTEIN

Group 67

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ABSTRACT

This note considers the problem of detecting the presence of pseudonoise waveforms. The detector is assumed to have no knowledge of the sequence providing the pseudorandom phase shift keying. It is found that the structure and performance of the optimum detector is nearly the same as a radiometer detector with proper prefiltering. In the special, but important, case when the received phase of a binary pseudonoise signal is constant over the detection interval it is found that the optimum detector performance is 3 dB better than the radiometric detector.

CONTENTS

I. INTRODUCTION AND CONCLUSIONS

In this note the structure of the optimum detector of pseudonoise waveforms is derived for three channel models. The three models are:

A. Phase known to the detector,

B. Phase unknown to the detector but constant over the detection interval,

and C. Phase random from chip to chip.

In each case the performance is analyzed for the practical situation of a low SNR per chip.

The received waveform, $r(t)$, is observed for N chips, each of duration T_c. Each chip waveform is of the form $\pm\sqrt{\frac{2E_c}{T_c}}\cos(\omega t+\theta_i)$ where θ_i is the phase on the i-th chip. The sign is unknown to the detector. In addition it is assumed that additive white Gaussian noise of single-sided density N_o is present. Statistics sufficient for the detector to make a decision consist of the results of correlating each chip against in-phase and quadrature references, $\sqrt{\frac{4}{N_oT_c}}\cos(\omega t)$ and $\sqrt{\frac{4}{N_oT_c}}\sin(\omega t)$. These will result in two statistics for each chip, r_{Ii} and r_{Qi} normalized to unity variance. Note that this correlation is equivalent to passing the received waveform through filters with $\frac{\sin(\pi f/T_c)}{(\pi f/T_c)}$ amplitude characteristics but 90^o offset phase characteristics and sampling the outputs every T_c seconds. While synchronization is assumed in each case, this is not a severe restriction. Clearly several time origins may be examined, one of which will be close to the results presented here. The false alarm rate would consequently rise - but an insignificant gain in required detector SNR will compensate for this.

The behavior of the optimum detector for binary pseudonoise waveforms with small E_c/N_o is found to be similar to a radiometric detector preceded by a signal spectrum matching filter. However, in cases of practical interest there is a 3-dB improvement in SNR compared to the radiometer. If the phase of the received signal is known, the detector SNR will be:

$$d^2 = \frac{2T}{(1/T_c)} \left(\frac{P_r}{N_o}\right)^2$$

where the total observation time is $T=NT_c$, and the received power is $P_r=E_c/T_c$. The false alarm probability and detection probabilities being given by the following trade-off equations:

$$p_f = Q(\gamma)$$

$$p_d = Q(\gamma-d)$$

where γ is a threshold.* If the phase is random from chip-to-chip d^2 is reduced by 3 dB. (A change in required (P_r/N_o) of only 1.5 dB.) The case of unknown but constant phase (which clearly must be between the two previous cases) has a d^2 essentially equal to the known-phase case.

It should be noted that higher order phase modulation (e.g., quadraphase) would be bounded in behavior by the known and random from chip-to-chip cases above.

II. KNOWN-PHASE MODEL

If θ_i is known, the r_{Ii} terms can be assumed to be in-phase with the received waveform. r_{Qi} can be ignored. The likelihood ratio, $\lambda(\vec{r})$, where $\vec{r}$

* $$Q(x) \triangleq \int_x^\infty \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx$$

refers to the entire set of r_{Ii} and r_{Qi} observed can be written as:

$$\lambda(\vec{r}) = \prod_N \frac{\frac{1}{2}\frac{1}{\sqrt{2\pi}}\left[e^{-(r_{Ii}-m)^2/2} + e^{-(r_{Ii}+m)^2/2}\right]}{\frac{1}{\sqrt{2\pi}}\, e^{-r_{Ii}^2/2}}$$

where $m = \sqrt{\frac{2E_c}{N_o}}$. The product is taken over the N chips under observation. Each numerator term is just the probability density function (pdf) of r_{Ii} which is Gaussian with unit variance and mean $\pm m$ with equal probability. Canceling common terms and factoring terms out of the product which are independent of $\vec{r}$ gives the simpler form:

$$\lambda(\vec{r}) = \text{const} \prod_N \left[e^{2r_{Ii}m} + e^{-2r_{Ii}m}\right] .$$

At this point, $\lambda(\vec{r})$ can be manipulated in two ways. If the product were expanded into a sum, there would be 2^N terms each of the form $e^{2m\Sigma \pm r_{Ii}}$ where the sum in the exponent would vary over all the 2^N ways that signs can be chosen to combine the r_{Ii} terms. This shows that the likelihood ratio for this composite hypothesis testing problem is the sum of the likelihood ratios for each of the possible 2^N received waveforms, a well-established fact but not a practical way of designing a detector.

A more interesting manipulation is the following:

$$\lambda(\vec{r}) = \text{const} \prod_N \cosh\,(2r_{Ii}m)$$

after taking the logarithm (neglecting constants):

$$L(\vec{r}) = \sum_N \ln\cosh\,(2r_{Ii}m)$$

which shows that 2^N filters need not be built. Indeed, $L(\vec{r})$ consists of a non-linear weighting of the matched filter outputs which are then summed. The sum, $L(\vec{r})$, would then of course be compared to a threshold.

The most interesting practical case is when m, $\sqrt{\frac{2E_c}{N_o}}$, is small. In this case, the approximation $\ln \cosh(x) \approx \frac{x^2}{2}$ can be used to show that (except for constants):

$$\boxed{L(\vec{r}) \approx \sum_N r_{Ii}^2 \; .}$$

Hence the optimum detector is a square-law or energy detector. However 'half' of the received energy is not included since it is known to be noise. A structure for the optimum receiver is shown in Fig. 1.

The impulse response of the filter shown is:

$$h_c(t) = \begin{cases} \sqrt{\frac{4}{N_o T_c}} \cos\omega t & 0 \le t < T_c \\ 0 & \text{elsewhere.} \end{cases}$$

The performance of this detector follows readily from noting that $L(\vec{r})$ is the sum of many independent variables and hence can be considered Gaussian. Thus all that is required for performance evaluation are the means and variances under noise-alone (hypothesis H_0) and signal-plus-noise (hypothesis H_1) conditions. For noise alone:

$$\begin{aligned} E\left[L(\vec{r})\right] &= N\, E\left[r_{Ii}^2\right] \\ &= N \end{aligned}$$

and

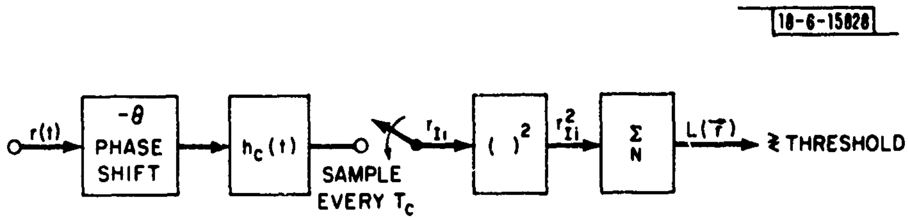

Fig. 1. Optimum detector-known phase.

$$\begin{aligned}
\mathrm{Var}\left[L(\vec{r})\right] &= N\ \mathrm{Var}\left[r_{Ii}^2\right] \\
&= N(E\left[r_{Ii}^4\right] - E^2\left[r_{Ii}^2\right]) \\
&= N\ (3-1) \\
&= 2N
\end{aligned}$$

which all follows from the fact that r_{Ii} is Gaussian, zero mean and normalized to unit variance. Assuming signal-plus noise:

$$\begin{aligned}
E\left[L(\vec{r})\right] &= N\ E\left[r_{Ii}^2\right] \\
&= N\ (1 + m^2) \\
\mathrm{Var}\ \ L(r) &= N\ \mathrm{Var}\left[r_{Ii}^2\right] \\
&= N\ (E\left[r_{Ii}^4\right] - E^2\left[r_{Ii}^2\right]) \\
&= N\ (m^4 + 6m^2 + 3 - (1 + m^2)^2) \\
&= N\ (4m^2 + 2)
\end{aligned}$$

which follow from the fact that r_{Ii} is a zero mean, Gaussian with unit variance added to $\pm m$. Since moments of r_{Ii}^2 are being computed the sign of m is irrelevant.

Hence when signal is present the mean of $L(\vec{r})$ increases by Nm^2, the variance remains 2N when terms of order m^2 are neglected. The "signal-to-noise ratio" of this detector, d^2, is then:

$$\begin{aligned}
d^2 &= \frac{(\Delta \text{ mean})^2}{\text{variance}} \\
&= \frac{Nm^4}{2} \\
&= 2N\ (\frac{E_c}{N_o})^2 .
\end{aligned}$$

Relating this to the received power, $P_r = E_c/T_c$ and the total observation time, $T = NT_c$ gives:

$$d^2 = \frac{2T}{(1/T_c)} \left(\frac{P_r}{N_o}\right)^2$$

which is 3 dB better than a radiometer of integration time T trying to detect a signal in bandwidth $W = 1/T_c$. The 3-dB gain comes about because of the known phase (as will be shown below). It should be noted that this detector actually uses the full signal energy by correlating with chip reference waveforms. A detector which consisted merely of a rectangular filter of bandwidth $1/T_c$ Hz followed by a sampler, squarer and adder would have about 2 dB poorer performance (i.e., d^2 reduced by 2 dB) since 1 dB of the signal power would be filtered out.

III. UNKNOWN, CONSTANT PHASE

The case when the θ_i are all equal (to θ), but unknown proceeds in a similar way. The likelihood ratio can be written as:

$$\lambda(\vec{r}) = \overline{\prod_N \left[\frac{\frac{1}{4\pi}\left[e^{-(r_{Ii}-m\cos\theta)^2/2-(r_{Qi}-m\sin\theta)^2/2} + e^{-(r_{Ii}+m\cos\theta)^2/2-(r_{Qi}+m\sin\theta)^2/2}\right]}{\frac{1}{2\pi}\, e^{-(r_{Ii}^2 + r_{Qi}^2)/2}} \right]}^{\theta}$$

$$= \text{const}\ \overline{\prod_N e^{m(r_{Ii}\cos\theta + r_{Qi}\sin\theta)} + e^{-m(r_{Ii}\cos\theta + r_{Qi}\sin\theta)}}^{\theta}$$

$$= \text{const}\ \overline{\prod_N \cosh\ m(r_{Ii}\cos\theta + r_{Qi}\sin\theta)}^{\theta}$$

where the whole expression must be averaged over θ (which is assumed to have a uniform distribution).

If the product is expanded into a sum before averaging, 2N terms will appear of the form:

$$\overline{e^{m\left[(\Sigma \pm r_{Ii}) \cos \theta + (\Sigma \pm r_{Qi}) \sin \theta\right]}}^{\theta}$$

where again the $\Sigma\pm$ notation refers to all combinations of sign used in the addition. Then, using the Bessel relationship,

$$\frac{1}{2\pi} \int_0^{2\pi} e^{a \cos \theta + b \sin \theta} \, d\theta = I_o \left(\sqrt{a^2+b^2}\right)$$

performing the average over θ will lead to a sum of 2^N terms of the form $I_o\left(m\sqrt{(\Sigma\pm r_{Ii})^2 + (\Sigma\pm r_{Qi})^2}\right)$. This again is the sum of the likelihood ratios for each possible 2^N waveforms after envelope detection. (The Bessel function argument is the envelope.)

The more interesting approach for small m is again to use the expansion for $\ell n \cosh x \approx \frac{x^2}{2}$ to show that

$$\lambda(\vec{r}) = \overline{\exp\left[\Sigma\, \ell n\, (\cosh\, (m(r_{Ii} \cos \theta + r_{Qi} \sin \theta)))\right]}^{\theta}$$

$$\approx \overline{\exp \frac{m^2}{2} \left[\Sigma(r_{Ii} \cos \theta + r_{Qi} \sin \theta)^2\right]}^{\theta}$$

$$= \overline{\exp \frac{m^2}{2} \left[(\Sigma r_{Ii}^2)\cos^2 \theta + (\Sigma r_{Qi}^2)\sin^2 \theta + 2(\Sigma r_{Ii} r_{Qi})\cos \theta \sin \theta\right]}^{\theta}$$

$$= \overline{\exp \frac{m^2}{2} (A \cos^2 \theta + B \sin^2 \theta + 2\, C \cos \theta \sin \theta)}^{\theta}$$

where A, B, C represent the three sums. The average over θ can be performed by integrating over $(0, 2\pi)$ which, after some trigonometry yields:

$$\lambda(\vec{r}) = e^{m^2(A+B)/4} \frac{1}{2\pi} \int_0^{2\pi} e^{m^2/2 \left[(A-B)/2 \cos 2\theta + C \sin 2\theta\right]} d\theta$$

$$= e^{m^2(A+B)/4} I_o\left(\frac{m^2}{2} \sqrt{\frac{(A-B)^2}{4} + C^2}\right)$$

where

$$A = \Sigma r_{Ii}^2$$

$$B = \Sigma r_{Qi}^2$$

$$C = \Sigma r_{Ii} r_{Qi}.$$

Note that the only approximation used to derive the above is that of small m. If it is now assumed that the argument of the Bessel function is large (greater than ≈ 3) its asymptotic expansion, $I_o(x) \sim \frac{1}{\sqrt{2\pi x}} e^x$, can be used. Ignoring the $\sqrt{x}$ term and other constants, then taking the logarithm of $\lambda(\vec{r})$ yields

$$L(\vec{r}) \approx \frac{(A+B)}{2} + \sqrt{\frac{(A-B)^2}{4} + C^2} \quad .$$

It will be presently shown that the approximation of the large argument is a good one. In any case, the above expression defines a good receiver structure even if not the optimum one. After some manipulation the receiver can be shown to be of the form shown in Fig. 2. The detector has an interesting interpretation. The top branch is essentially a radiometer preceded by a filter matched to the signal spectrum. The output of the squarer also contains double-frequency energy. The bottom, double branch, is used to detect double-frequency energy coherent over the measuring interval. It is similar to the structure of a squaring loop which utilizes the fact that squaring a biphase signal plus noise strips off the biphase modulation. As will be shown below,

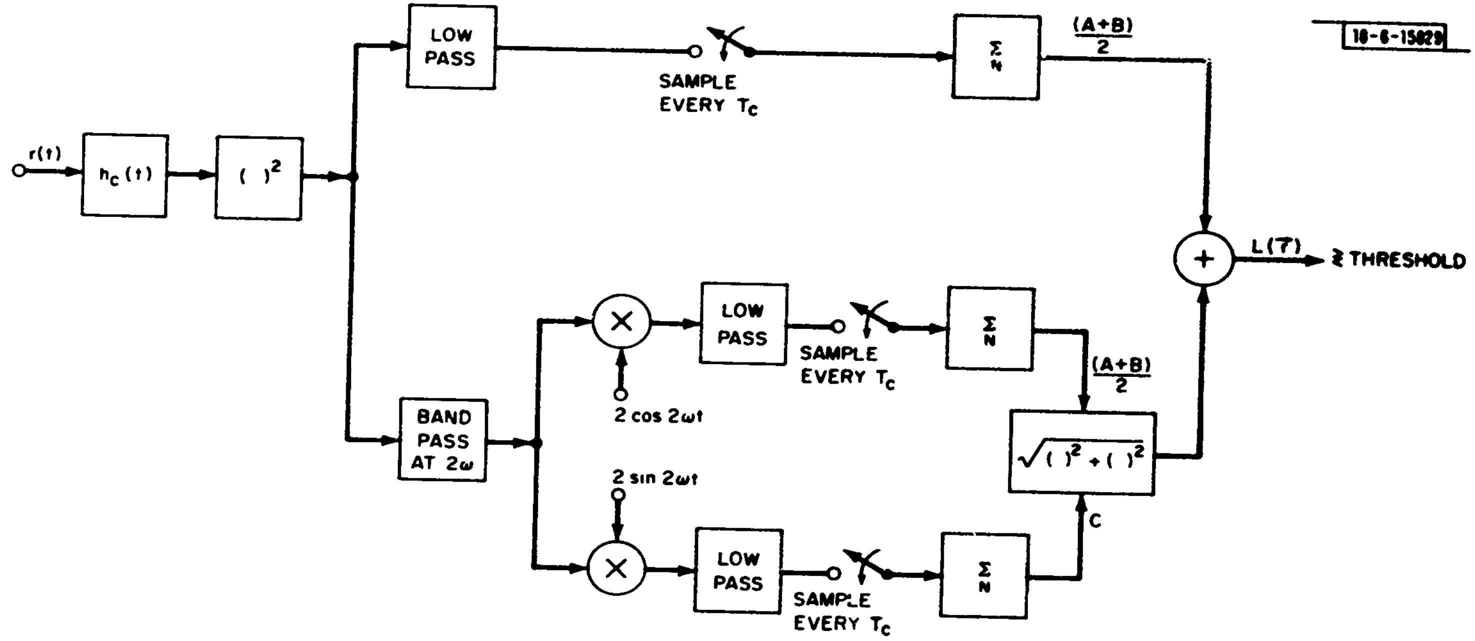

Fig. 2. Optimum detector-constant but unknown phase.

the low-pass and double-frequency noise terms are independent, thus it is not surprising that both bands should be used for detection.

The analysis of this detector is unfortunately rather tedious. Hence only a sketch of the derivation will be given. The assumption that will be made is that A, B and C are jointly Gaussian since each is the sum of many independent terms. The following second-order statistics can be derived in a straight-forward manner (assuming hypothesis H_1 and $m_I = m \cos \theta$, $m_Q = m \sin \theta$):

$$E[A] = N(1+m_I^2) \qquad \text{Var}[A] = N(4m_I^2 + 2)$$
$$E[B] = N(1+m_Q^2) \qquad \text{Var}[B] = N(4m_Q^2 + 2)$$
$$E[C] = Nm_I m_Q \qquad \text{Var}[C] = N(1+m_I^2 + m_Q^2).$$

The terms appearing in $L(\vec{r})$ are A+B, A-B and C which will also be Gaussian. It can be shown that these three variables are uncorrelated (hence independent) and that $\text{Var}[A+B] = \text{Var}[A-B] = N(4+4m^2)$. The mutual lack of correlation shows that the double-frequency noise terms are independent of the low-frequency terms in Fig. 2.

$L(\vec{r})$ is now seen to be the sum of two independent terms, $\frac{A+B}{2}$ which is Gaussian, with mean N and variance N, and $\sqrt{\frac{(A-B)^2}{4} + C^2}$ which has a Raleigh distribution with mean $\sqrt{\frac{\pi}{2}}N$ under H_o.* Under H_1 the second term has a Rician distribution. By a direct calculation it can be shown that the p.d.f. for $L(\vec{r})$ assuming H_o is:

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* Note that the Bessel function argument has mean $\sqrt{\frac{\pi N m^4}{8}}$ which is large enough to justify the large-argument approximation.

$$p_L(\ell) = \frac{1}{2\sqrt{N}}\left[\frac{(\ell-N)Q((N-\ell)/\sqrt{2})}{\sqrt{2N}}\, e^{-(\ell-N)^2/4N} + \frac{e}{\sqrt{2\pi N}}^{-(\ell-N)^2/2N}\right].$$

Since one is most always interested in low false alarm probabilities, it is the tail of this distribution which is of most importance. This is the region where $\ell >> N$ and the approximation $Q(x) = 1 - \frac{1}{2} e^{-x^2/2}$ for $x << 0$ can be used. Keeping only the largest term gives:

$$p_L(\ell) \approx \frac{(\ell-N)}{2\sqrt{2}N} e^{-(\ell-N)^2/4N}$$

from which it follows that the false alarm probability is:

$$Pr(L>\ell) \approx \frac{1}{\sqrt{2N}} e^{-(\ell-N)^2/4N} .$$

Thus the false alarm probability behaves as if the test statistic, L, was Gaussian with mean N and variance 2N.

Under H_1, the mean of $\frac{A+B}{2}$ increases by $\frac{Nm^2}{2}$ while the variance stays essentially unchanged at N. The square-root term will be essentially Gaussian with mean $\frac{Nm^2}{2}$ and variance N as long as $\frac{Nm^4}{4} >> 1$ which, it will be shown, is the condition for good signal detectability. Thus under H_1, L is again Gaussian with variance 2N with the new mean $N + Nm^2$. It follows that d^2 can be defined in the same way as in the known-phase case and will be:

$$d^2 = \frac{Nm^4}{2}$$

or

$$d^2 = \frac{2T}{(1/T_c)} \left(\frac{P_r}{N_o}\right)^2 \quad \text{for} \quad d^2 >> 1.$$

This result is identical to that for known θ which is reasonable in retrospect

since it can be shown that the received phase can be estimated with standard deviation

$$\sigma_\theta \approx \frac{1}{\sqrt{N}\, m^2}$$

$$= \frac{1}{\sqrt{2}\, d} \text{ radians.}$$

Hence the phase estimate will be quite accurate for large d making this case equivalent to the case of known θ. (The phase estimate with this accuracy is given by:

$$\hat{\theta} = \frac{1}{2} \arctan \frac{2C}{A-B} \quad .)$$

IV. PHASE RANDOM FROM CHIP-TO-CHIP

In this case the θ_i are assumed to be uniformly distributed from 0 to 2π and independent from chip-to-chip. The likelihood ratio is similar to that of the previous case, except here the averaging over θ_i is done for each term in the product before multiplying:

$$\lambda(\vec{r}) = \text{const} \prod_N \overline{\cosh\ m(r_{Ii} \cos \theta_i + r_{Qi} \sin \theta_i)}^{\theta_i}$$

$$= \text{const} \prod_N I_o(m\sqrt{r_{Ii}^2 + r_{Qi}^2}) \quad .$$

In this case it is seen that there is no structure which can take advantage of the 2^N possible transmitted waveforms since they are completely scrambled by the random phase shifts. Taking logarithms and neglecting constants:

$$L(\vec{r}) = \sum_N \ell n\ I_o(m\sqrt{r_{Ii}^2 + r_{Qi}^2}) \quad .$$

For small m the sufficient statistic can be simply:

$$\boxed{L(\vec{r}) \approx \sum_N r_{Ii}^2 + r_{Qi}^2}$$

which is similar to the known-θ case, but now both in-phase and quadrature components are included. The receiver structure consists of just the top branch of the receiver in the previous case. The Gaussian approximation for $L(\vec{r})$ will again be involved with the following means and variances:

H_0, noise alone: $E\left[L(\vec{r})\right] = 2N$

$\mathrm{Var}\left[L(\vec{r})\right] = 4N$

H_1, signal plus noise: $E\left[L(\vec{r})\right] = N(2+m^2)$

$\mathrm{Var}\left[L(\vec{r})\right] = N(4 + 4m^2)$.

Hence for this case:

$$d^2 = \frac{Nm^4}{4}$$

$$= \frac{T}{1/T_c}\left(\frac{P_r}{N_o}\right)^2$$

which is essentially the same as a radiometer detector with proper pre-filtering and only 3 dB worse in terms of d^2 than the known-phase case. (The difference in terms of P_r/N_o will be only 1.5 dB.)